

The "freely" falling two-level atom in a running laser wave

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Abstract

The time evolution of a two-level atom which is simultaneously exposed to the field of a running laser wave and a homogeneous gravitational field is studied. The result of the coupled dynamics of internal transitions and center-of-mass motion is worked out exactly. Neglecting spontaneous emission and performing the rotating wave approximation we derive the complete time evolution operator in an algebraical way by using commutation relations. The result is discussed with respect to the physical implications. In particular the long time and short time behaviour is physically analyzed in detail. The breakdown of the Magnus perturbation expansion is shown.

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1 Introduction

Over the last ten years the manipulation of neutral atoms by laser light was dramatically improved and has led to the new field of atom optics (see Ref. [1] and references therein). It includes in particular atomic interferometry and laser cooling of atoms, i.e., the preparation of a dense cloud of atoms with a narrow momentum distribution. The width of the atomic velocity distribution can be made as small as 1 cm/s [2]. This is of interest for atomic interferometry because the possibility to use slow atom beams enlarges the phase shifts caused by a broad class of external potentials [3]. It is obvious that for atoms moving with a velocity of a few centimeters or meters per second for a time period of several milliseconds or more the influence of the earth's acceleration becomes important and cannot be neglected. For this reason it is of interest to study the time evolution of an atom moving in the gravitational field of the earth and the field of a laser beam.

This time evolution is the subject of this paper. In order to have a clear theoretical model which is exactly solvable we restrict to the case of a two-level atom moving in a running laser wave. Despite its relative simplicity the two-level model is well suited for the description of certain experiments such as Ramsey spectroscopy [4] or atomic interferometry [5, 6]. But this exact solvable model is not only accessible to experimental investigation. It may also serve as a reference for more complicated atoms with more than two levels which cannot be solved rigorously. Numerical methods used in this context can be tested with the two-level model. Another important advantage is the detailed physical discussion which can (and will) be made using exact solutions. The effect of every term in the Hamiltonian can easily be identified.

We describe the dynamical behaviour of the two-level atom under the simultaneous influence of the running laser wave and the earth's acceleration in the Schrödinger picture by means of the unitary time evolution operator

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle . \quad (1)$$

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which is in our case a 2×2 matrix. It is methodically essential for the following that we follow an algebraical approach to derive $U(t)$. This approach makes only use of commutation relations and turns out to be particularly transparent as far as a continuous compelling physical interpretation is concerned. Because of its independence of the initial state $|\psi_0\rangle$ the generality of the approach is evident. Moreover, it turns out that the resulting exact expressions for $U(t)$ are very compact. We mention that the algebraic method has already been used in the context of atomic interferometry to calculate the phase shift induced by external potentials including gravity for very general interferometer geometries [7]. The gravitational phase shift was also calculated by Bordé [8] with a similar method.

To get a clear picture of what is going to happen it may be useful to refer to the equivalence principle. It states that the influence of a homogeneous gravitational field on the atom moving in a laser wave can be simulated by constant acceleration. This means that the following situation is physically equivalent to our original set up: A two-level atom is at rest or moving with constant velocity relative to an inertial system. The laboratory with the laser attached to it moves with constant acceleration. The consequence is that the laser wave reaches the atom with Doppler shifted frequency. Because of the acceleration this shift changes in time. It acts as a time dependent detuning. The internal transitions of the atom in the limit of vanishing acceleration and accordingly constant Doppler shift are essentially known. For not too large detuning they are described as Rabi oscillations. In our case, because of the time dependent detuning, the internal behaviour of the atom in time must be worked out anew. This is the central task of this paper. Note that there is no further influence of gravity on the internal dynamics of the atom, because the fact that reference is made to an accelerated reference frame can have no measurable consequences in this regard.

Turning to the center-of-mass motion of the atom we know that in the rotating wave approximation the transitions between the two internal states are related to the absorption or emission of one photon. The corresponding energy and momentum transfer by recoil must show up in the final state $|\psi(t)\rangle$. It is contained in a simple way in the transition matrix elements $U_{12}(t)$ and $U_{21}(t)$.

Finally, there remains the motion of the atom relative to the accelerated reference frame (or from the other point of view, the free fall in the gravitational field). We will separate this center-of-mass motion from the internal dynamics and the recoil effects in factorizing $U(t)$ with $\exp(-iH_{c.m.}t/\hbar)$ as a factor on the left. $H_{c.m.}$ is the center-of-mass part of the Hamiltonian. Clearly this factor drops out if one calculates the probability of finding the atom in the excited state. This probability is not related to the actual position and momentum of the atom. It is influenced by gravity and acceleration only via the fact that the atom registers the laser wave during the time t with a time dependent Doppler shift. As stated above, the additional fact that the description refers to an accelerated reference frame does not influence internal atomic processes.

The paper is organized as follows. In section 2 we will introduce the model Hamiltonian and perform a unitary transformation to get a time-independent Hamiltonian. The exact result for the time evolution operator $U(t)$ will be derived in section 3. Sections 4 and 5 are devoted to a small influence of gravity and the long time limit, respectively. A surprising and not commonly known breakdown of perturbation theory will be demonstrated in section 6. In section 7 we will conclude the main results and discuss possible applications.

2 The model

We suppose the Hamiltonian of the two-level atom to be of the form

$$H = \begin{pmatrix} E_e - i\gamma_e/2 & \\ & E_g - i\gamma_g/2 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} H_{c.m.} - \vec{d} \cdot \vec{E} \quad , \quad (2)$$

where E_e (E_g) and γ_e (γ_g) are the energy and the decay rate of the excited state $|\psi_e\rangle$ and the ground state $|\psi_g\rangle$, respectively. Although we have inserted the decay factors in a phenomenological way, a more complete description of the spontaneous emission would require a density matrix approach. For this reason the validity of our model is restricted to times shorter than the life time of the states, but this may be quite long for metastable states. For convenience, we will set $\gamma_i = 0$ in the calculation. This is no loss of generality since they can be reintroduced by replacing E_i by $E_i - i\gamma_i/2$ in all expressions.

$$H_{c.m.} = \frac{\vec{p}^2}{2M} - M\vec{a} \cdot \vec{x} \quad (3)$$

is the center-of-mass part of the Hamiltonian with \vec{x} , \vec{p} , and M being the position and momentum operator and the mass of the atom. \vec{a} denotes the constant gravitational acceleration acting on the atom. The interaction with the running laser wave is modeled by the dipole coupling

$$\vec{d} \cdot \vec{E} = \hbar\Omega \cos[\omega_L t - \vec{k} \cdot \vec{x} + \varphi] \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad (4)$$

where $\Omega := \vec{d} \cdot \vec{E}_0 / \hbar$ is Rabi's frequency. \vec{E}_0 is the amplitude of the laser wave, ω_L its frequency, φ its phase, and \vec{k} its wave vector. \vec{d} is the dipole moment of the two-level atom.

In this form the Hamiltonian depends explicitly on the time t . To get rid of this dependence thus making the dynamics simple we make a unitary transformation with the operator

$$O(t) = \begin{pmatrix} \exp[i(\omega_L t - \vec{k} \cdot \vec{x} + \varphi)/2] & \\ & \exp[-i(\omega_L t - \vec{k} \cdot \vec{x} + \varphi)/2] \end{pmatrix} \quad (5)$$

In the rotating wave approximation, i.e., after neglecting all terms oscillating with the frequency $2\omega_L$, we find for the transformed Hamiltonian $\tilde{H} = OHO^{-1} - i\hbar O\dot{O}^{-1}$ the expression

$$\tilde{H} = \begin{pmatrix} E_e & \\ & E_g \end{pmatrix} + \left\{ H_{c.m.} + \frac{\hbar\delta}{4} \right\} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \frac{\hbar}{2} \{ \omega_L - \hat{D} \} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{\hbar\Omega}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad (6)$$

where we have introduced the well known recoil shift

$$\delta := \frac{\hbar\vec{k}^2}{2M}. \quad (7)$$

The operator

$$\hat{D} := \frac{1}{M} \vec{p} \cdot \vec{k} \quad (8)$$

is crucial for the following calculations and their physical implications. For an atom moving with velocity $\vec{v} = \vec{p}/M$ it can be written as $\vec{v} \cdot \vec{k}$. Hence this operator represents the Doppler shift of the laser frequency in the rest frame of the atom. We will call it the *Doppler operator*. Strictly speaking we are in this context only allowed to argue with reference to the transformed momentum operator

$$\widetilde{\vec{p}\mathbf{1}} = O\vec{p}\mathbf{1}O^{-1} = \vec{p}\mathbf{1} + \frac{1}{2}\hbar\vec{k}\sigma_3 \quad (9)$$

where $\mathbf{1}$ is the unit matrix in two dimensions and σ_i are the Pauli matrices. But with Eq. (9) it is easy to see that the term containing the Doppler operator can be written as

$$\hat{D}\sigma_3 = \widetilde{\hat{D}\sigma_3} - \delta\mathbf{1} \quad (10)$$

so that the interpretation is the same apart from a trivial shift of the energy eigenvalues.

In addition to the fact that it implies a time independent Hamiltonian, the unitary transformation (5) has two important consequences. The momentum transfer to the center-of-mass motion related to the absorption or emission of a photon has been absorbed in $O(t)$. That the laser causes internal transitions is reflected as usual by the Ω -term. The Hamiltonian (6) shows very clearly that after separation of the momentum transfer the term containing the Doppler operator and therefore the operator p is the only one which couples the internal degrees of freedom to the center-of-mass motion in a nontrivial way.

3 The exact time evolution operator

The main advantage of the unitary transformation (5) was to get rid of the explicit time dependence in the Hamiltonian (after the rotating wave approximation). As a consequence the new evolution operator $\tilde{U}(t)$ can simply be determined by calculating

$$O(t)U(t)O^{-1}(0) = \tilde{U}(t) = \exp[-it\tilde{H}/\hbar] =: e^{A+B} \quad (11)$$

with

$$\begin{aligned} A &:= Q\mathbf{1} \\ B &:= P\sigma_3 + R\sigma_1 \end{aligned} \quad (12)$$

and

$$\begin{aligned} Q &:= \frac{-it}{\hbar} \left\{ H_{c.m.} + \frac{1}{2} [E_e + E_g + \hbar\delta/2] \right\} \\ P &:= \frac{it}{2} \{\Delta - \hat{D}\} \\ R &:= \frac{it}{2} \Omega . \end{aligned} \quad (13)$$

The operators A, B, P, Q , and R are introduced to clarify the mathematical structure of the calculation below. In P the quantity

$$\Delta := \omega_L - \omega_{eg} \quad \text{with} \quad \omega_{eg} := \frac{E_e - E_g}{\hbar} \quad (14)$$

denotes the detuning of the laser frequency with respect to the atomic transition.

To separate the free fall of the atom's center of mass we factorize $\tilde{U}(t)$ according to

$$\tilde{U}(t) = e^A W(t) . \quad (15)$$

To know the complete time development of the two-level atom it now remains to determine the operator $W(t)$. It contains the influence of the gravitation on the internal dynamics. It is easy to recognize its structure. The factorization (15) will introduce in $W(t)$ the commutator $[A, B]$ and therefore

$$[Q, P] = \frac{t^2}{2\hbar} [\hat{D}, H_{c.m.}] = \frac{i}{2} \vec{k} \cdot \vec{a} t^2 . \quad (16)$$

This is a c-number. The second equation shows that for $\vec{k} \cdot \vec{a} \neq 0$ gravity causes a time dependent Doppler shift. This is the central physical effect. We introduce

$$\hat{D}_t := \hat{D} + \vec{k} \cdot \vec{a} t \quad (17)$$

(it may also be written as $\vec{k} \cdot (\vec{p}/M + \vec{a}t)$). The relevant time scale is

$$\tau_a := \frac{1}{\sqrt{|\vec{k} \cdot \vec{a}|}} . \quad (18)$$

For an optical laser with $|\vec{k}| \approx 10^7 \text{ m}^{-1}$ and the earth's acceleration ($|\vec{a}| = 9.81 \text{ m/s}^2$) τ_a is about 10^{-4} seconds. Introducing

$$\zeta := \text{sgn}(\vec{k} \cdot \vec{a}) \quad (19)$$

we can rewrite Eq. (16) as

$$[Q, P] = \frac{i}{2} \zeta \frac{t^2}{\tau_a^2} . \quad (20)$$

To work out $W(t)$ of Eq. (15) we employ a method which was used by Lutzky [9] in the context of the Baker-Campbell-Hausdorff formula. We consider the operator

$$G(\lambda) := \exp[\lambda(A + B)] =: e^{\lambda A} W(\lambda) . \quad (21)$$

and restrict to $\lambda = 1$ at the end. Differentiation of Eq. (21) with respect to λ leads to the differential equation

$$\frac{dW}{d\lambda} = (e^{-\lambda A} B e^{\lambda A}) W(\lambda) \quad (22)$$

with the initial condition $W(\lambda = 0) = \mathbf{1}$. Using the identity

$$e^{-\lambda A} B e^{\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} K_n \quad , \quad K_0 := B \quad , \quad K_{n+1} := [K_n, A] \quad (23)$$

which holds for any two operators A, B one can simplify Eq. (22) in our case to

$$\frac{dW}{d\lambda} = (B - \lambda[Q, P]\sigma_3) W(\lambda) . \quad (24)$$

It may be written as a matrix equation,

$$\begin{pmatrix} dW_{11}/d\lambda & dW_{12}/d\lambda \\ dW_{21}/d\lambda & dW_{22}/d\lambda \end{pmatrix} = \begin{pmatrix} P - \lambda[Q, P] & R \\ R & -P + \lambda[Q, P] \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} . \quad (25)$$

This is an operator-valued system of differential equations, but it contains only commuting operators (since $[Q, P]$ is a c-number) so that we can treat it as an ordinary differential equation.

Inserting the equation for $dW_{11}/d\lambda$ (for $dW_{22}/d\lambda$) into the equation for $dW_{21}/d\lambda$ (for $dW_{12}/d\lambda$) one arrives at

$$\begin{aligned} \frac{d^2 W_{11}}{d\lambda^2} &= \{R^2 - [Q, P] + (\lambda[Q, P] - P)^2\} W_{11} \\ \frac{d^2 W_{22}}{d\lambda^2} &= \{R^2 + [Q, P] + (\lambda[Q, P] - P)^2\} W_{22} . \end{aligned} \quad (26)$$

After the introduction of the parameter

$$\theta := \frac{R^2}{2[Q, P]} = i \frac{\Omega^2}{4\vec{k} \cdot \vec{a}} \quad (27)$$

and the change to the variable $y := (\lambda[Q, P] - P)\sqrt{2/[Q, P]}$ Eq. (26) becomes

$$\frac{d^2 W_{11}}{dy^2} = \left\{ \frac{y^2}{4} + \theta - \frac{1}{2} \right\} W_{11}(y) \quad (28)$$

$$\frac{d^2 W_{22}}{dy^2} = \left\{ \frac{y^2}{4} + \theta + \frac{1}{2} \right\} W_{22}(y) \quad (29)$$

with the initial conditions $W_{11}(\lambda = 0) = W_{22}(\lambda = 0) = 1$ and

$$\left. \frac{dW_{11}}{dy} \right|_{\lambda=0} = - \left. \frac{dW_{22}}{dy} \right|_{\lambda=0} = \frac{P}{2} \sqrt{\frac{2}{[Q, P]}} = \frac{i}{2\sqrt{i}\zeta} \tau_a \hat{\Delta}_0 \quad (30)$$

($\hat{\Delta}_0 := \Delta - \hat{D}$). We used the opportunity to introduce the operator of the time dependent Doppler shifted detuning

$$\hat{\Delta}_t := \Delta - \hat{D}_t = \Delta - \hat{D} - \vec{k} \cdot \vec{a}t \quad (31)$$

in which the effect of acceleration is already included.

The solution of Eq. (28) and Eq. (29) is a linear combination of parabolic cylinder functions. All relations between these functions used in the following are taken from chapter 19 of Ref. [10]. The standard solutions are given by

$$U(\alpha, y) = \sqrt{\pi} e^{-y^2/4} \left\{ \frac{{}_1F_1(\alpha/2 + 1/4, 1/2, y^2/2)}{2^{\alpha/2+1/4} \Gamma(\alpha/2 + 3/4)} - \frac{y {}_1F_1(\alpha/2 + 3/4, 3/2, y^2/2)}{2^{\alpha/2-1/4} \Gamma(\alpha/2 + 1/4)} \right\} \quad (32)$$

and

$$V(\alpha, y) = \frac{\Gamma(\alpha + 1/2)}{\pi} \{ \sin(\pi\alpha) U(\alpha, y) + U(\alpha, -y) \} \quad (33)$$

where ${}_1F_1(\alpha, \beta, y)$ is the confluent hypergeometric function. For W_{11} we have to set $\alpha = -1/2 + \theta$, and for W_{22} we have $\alpha = 1/2 + \theta$. The linear coefficients can be derived from the initial conditions. By using the Wronskian relation

$$U \frac{dV}{dy} - \frac{dU}{dy} V = \sqrt{\frac{2}{\pi}} \quad (34)$$

as well as

$$\begin{aligned} \frac{dU}{dy}(\alpha, y) + \frac{1}{2}yU(\alpha, y) + (\alpha + \frac{1}{2})U(\alpha + 1, y) &= 0 \\ \frac{dV}{dy}(\alpha, y) + \frac{1}{2}yV(\alpha, y) - V(\alpha + 1, y) &= 0 \end{aligned} \quad (35)$$

one deduces (and from now on we set $\lambda = 1$)

$$\begin{aligned} W_{11}(\vec{p}) &= \sqrt{\frac{\pi}{2}} \left\{ V(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) U(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) + \theta U(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) V(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) \right\} \\ W_{12}(\vec{p}) &= \sqrt{\frac{\pi}{2}} \theta \left\{ U(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) V(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) - V(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) U(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) \right\} \\ W_{21}(\vec{p}) &= \sqrt{\frac{\pi}{2}} \theta \left\{ U(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) V(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) - V(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) U(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) \right\} \\ W_{22}(\vec{p}) &= \sqrt{\frac{\pi}{2}} \left\{ \theta V(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) U(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) + U(-\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_0) V(\tfrac{1}{2} + \theta, \tfrac{-i}{\sqrt{i\zeta}}\tau_a \hat{\Delta}_t) \right\} \end{aligned} \quad (36)$$

The time dependence is contained in $\hat{\Delta}_t$.

With Eq. (36) the total operator $\tilde{U}(t)$ is given by

$$\tilde{U}(t) = \exp \left\{ -it \left(\frac{E_e + E_g}{2\hbar} + \frac{\delta}{4} + \frac{1}{\hbar} H_{c.m.} \right) \right\} \begin{pmatrix} W_{11}(\vec{p}) & W_{12}(\vec{p}) \\ W_{21}(\vec{p}) & W_{22}(\vec{p}) \end{pmatrix}. \quad (37)$$

It remains to cancel the initial unitary transformation $O(t)$ in Eq. (11) to obtain the exact expression for the time development operator:

$$\begin{aligned} U(t) &= O^{-1}(t) \tilde{U}(t) O(0) \\ &= \exp \left\{ -it \left(\frac{E_e + E_g}{2\hbar} + \frac{\delta}{2} + \frac{1}{\hbar} H_{c.m.} \right) \right\} \begin{pmatrix} e^{-it(\omega_L - \hat{D})/2} e^{i\vec{k} \cdot \vec{a} t^2/4} & \\ & e^{it(\omega_L - \hat{D})/2} e^{-i\vec{k} \cdot \vec{a} t^2/4} \end{pmatrix} \times \\ &\quad \begin{pmatrix} W_{11}(\vec{p} - \hbar\vec{k}/2) & W_{12}(\vec{p} - \hbar\vec{k}/2) e^{i(\vec{k} \cdot \vec{x} - \varphi)} \\ W_{21}(\vec{p} + \hbar\vec{k}/2) e^{-i(\vec{k} \cdot \vec{x} - \varphi)} & W_{22}(\vec{p} + \hbar\vec{k}/2) \end{pmatrix}. \end{aligned} \quad (38)$$

This is the main result of the paper. The argument $\vec{p} \pm \hbar\vec{k}/2$ in the operators W_{ij} denotes that the operator \vec{p} has to be replaced by this expression wherever it occurs in W_{ij} of Eq. (36).

For practical calculations it is useful to rewrite the factor $\exp\{-itH_{c.m.}/\hbar\}$ in Eq. (38) with the aid of the Baker-Campbell-Hausdorff formula. Doing so one arrives at

$$\exp \left\{ \frac{-it}{\hbar} \frac{\vec{p}^2}{2M} \right\} \exp \left\{ \frac{it}{\hbar} M \vec{a} \cdot \vec{x} \right\} \exp \left\{ \frac{it^2}{2\hbar} \vec{a} \cdot \vec{p} \right\} \exp \left\{ \frac{it^3}{3\hbar} M \vec{a}^2 \right\}. \quad (39)$$

This form is especially convenient for the interpretation. The first exponential in Eq. (39) describes the ordinary kinetic energy of the atom. The second term describes the momentum gain $M\vec{a}t$ of the accelerated atom since the operator $\exp(i\vec{k} \cdot \vec{x})$ for any c-number vector \vec{k} amounts in adding $\hbar\vec{k}$ to the momentum. The third term is responsible for the displacement of the atom by the amount $\vec{a}t^2/2$, and the last term is a time dependent c-number phase factor which corrects the "error" that we have made in writing all factors of $H_{c.m.}$ in separated exponentials.

Turning to the discussion of the result (38) we note that the momentum transfer to the atom related to the absorption or emission of a photon can be read off from U_{12} and U_{21} (see the exponential factor on the right). The time dependence of W goes solely back to the time dependent Doppler operator \hat{D}_t contained in $\hat{\Delta}_t$ which modifies the detuning. \hat{D}_t introduces thereby a dependence from the center-of-mass state of the atom via the momentum operator. Those who are familiar with the evolution of a two-level atom in a running laser wave may miss the recoil shift δ in $\hat{\Delta}_t$ which normally comes with the detuning [4, 3]. It enters $\hat{\Delta}_t$ only after the retransformation of \tilde{U} in Eq. (38). The replacement of \vec{p} by $\vec{p} \pm \hbar \vec{k}/2$ in $W_{ij}(\vec{p} \pm \hbar \vec{k}/2)$ amounts in replacing $\hat{\Delta}_t$ in by $\hat{\Delta}_t \mp \delta$ so that the missing recoil shift is reproduced. Rabi's frequency Ω appears in the parameter θ in the form $\Omega \tau_a$. It grows with the intensity of the laser wave in relation to the characteristic time τ_a .

An instructive consistency check is to turn off the laser by setting Ω and hence θ to zero. In this case the freely falling two-level atom should be recovered. $\theta = 0$ implies immediately $W_{12} = W_{21} = 0$. For the evaluation of the W_{ii} it is necessary to use the identity [10]

$$U(-1/2, y) = e^{-y^2/4} \quad , \quad V(1/2, y) = \sqrt{\frac{2}{\pi}} e^{y^2/4} . \quad (40)$$

Inserting this into Eq. (36) and combining the result with Eq. (38) leads to

$$U(t) = \exp \left\{ \frac{-it}{\hbar} H_{c.m.} \right\} \begin{pmatrix} e^{-iE_e t/\hbar} & \\ & e^{-iE_g t/\hbar} \end{pmatrix} \quad (41)$$

as was to be expected. The first term describes the free fall of the atom, and the second term contains the internal oscillations. By using Eq. (40) it is also possible to derive an expansion of Eq. (38) for small Ω . But since the result contains various combinations of Error functions and is not much better to interpret as the complete result we will omit it here.

We will close this section with a mathematical note. In the definition of $\sqrt{i\zeta}$ the square root of a complex factor appears. Since \sqrt{z} for complex z is a multivalued function we should decide which branch has to be taken. Fortunately, this is not a big problem since the variable y in Eqs. (32) and (33) (and hence $\sqrt{i\zeta}$ in Eq. (37)) is squared in the argument of the function ${}_1F_1$. There is only one linear factor of y in Eq. (32) which can give an additional sign.

4 The limit of small gravitational influence

We now consider the limit of very large τ_a , or more precisely small $[Q, P]$. The results will thereby be valid for all times t with $t \ll \tau_a$ so that the atom will be able to perform many Rabi oscillations. The physical meaning of the limit is quite clear. Very large τ_a means very small $\vec{k} \cdot \vec{a}$, i.e., the momentum transfer from the laser beam to the atom is only slightly altered by the acceleration because the latter is very small or nearly perpendicular to the laser beam. The technical details of this limit are somewhat involved and are explained in Appendix A. The resulting expansion of $W_{ij}(\vec{p})$ up to linear powers of $\vec{k} \cdot \vec{a}$ is

$$\begin{aligned} W_{11}(\vec{p}) &= \cos(\hat{\omega}t/2) + i \frac{\hat{\Delta}_0}{\hat{\omega}} \sin(\hat{\omega}t/2) + \\ &\quad \frac{i}{4} \vec{k} \cdot \vec{a} t^2 \left\{ - \frac{\hat{\Delta}_0^2}{\hat{\omega}^2} \left[\cos(\hat{\omega}t/2) - \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right] - i \frac{\hat{\Delta}_0}{\hat{\omega}} \sin(\hat{\omega}t/2) - \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right\} \\ W_{12}(\vec{p}) &= \frac{i\Omega}{\hat{\omega}} \sin(\hat{\omega}t/2) - \frac{1}{2} \vec{k} \cdot \vec{a} t \frac{\Omega}{\hat{\omega}^2} \left(1 + \frac{it}{2} \hat{\Delta}_0 \right) \left\{ \cos(\hat{\omega}t/2) - \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right\} \\ W_{21}(\vec{p}) &= \frac{i\Omega}{\hat{\omega}} \sin(\hat{\omega}t/2) + \frac{1}{2} \vec{k} \cdot \vec{a} t \frac{\Omega}{\hat{\omega}^2} \left(1 - \frac{it}{2} \hat{\Delta}_0 \right) \left\{ \cos(\hat{\omega}t/2) - \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right\} \\ W_{22}(\vec{p}) &= \cos(\hat{\omega}t/2) - i \frac{\hat{\Delta}_0}{\hat{\omega}} \sin(\hat{\omega}t/2) + \\ &\quad \frac{i}{4} \vec{k} \cdot \vec{a} t^2 \left\{ \frac{\hat{\Delta}_0^2}{\hat{\omega}^2} \left[\cos(\hat{\omega}t/2) - \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right] - i \frac{\hat{\Delta}_0}{\hat{\omega}} \sin(\hat{\omega}t/2) + \frac{\sin(\hat{\omega}t/2)}{\hat{\omega}t/2} \right\} \end{aligned} \quad (42)$$

Here we have defined the frequency operator

$$\hat{\omega} := \sqrt{\Omega^2 + (-\hat{\Delta}_0)^2} \quad (43)$$

which incorporates the well known fact that the frequency of the Rabi oscillations is altered when the laser frequency is detuned versus the atomic transition frequency. Note that also the effect of the Doppler shift is included in $\hat{\Delta}_0$. The additional "−" sign before $\hat{\Delta}_0$ indicates that the negative branch of the square root has to be taken if Ω is set equal to zero.

The analysis of Eq. (42) is relatively simple. First, it is easy to show that for $\vec{k} \cdot \vec{a} = 0$ Eq. (42) describes the ordinary Rabi oscillations of a two level atom in a running laser wave. To see this we solve Eq. (24) with $[Q, P] = 0$ which corresponds to $\vec{k} \cdot \vec{a} = 0$. The obvious solution is

$$W = \exp\{\lambda B\} . \quad (44)$$

Since in the operator $B = P\sigma_3 + R\sigma_1$ the operators P and R commute one can apply the formula

$$\exp\{-i\vec{s} \cdot \vec{\sigma}\} = \mathbf{1} \cos(\sqrt{s^2}) - i \frac{\vec{s} \cdot \vec{\sigma}}{\sqrt{s^2}} \sin(\sqrt{s^2}) \quad (45)$$

to reproduce just the \vec{a} -independent part of Eq. (42). This result describes in operator form what has been obtained in previous calculations [4, 3] where the time evolution was derived (for certain wave packets) in momentum space. It is interesting to see the effect of the Doppler shift on the atomic evolution. It not only alters the frequency $\hat{\omega}$ of the Rabi floppings. A large Doppler shift also damps the transition probability because of the $1/\hat{\omega}$ dependence of the matrix elements $W_{12}(\vec{p})$ and $W_{21}(\vec{p})$, and it is (together with the detuning Δ) responsible for the imaginary part of W_{11} and W_{22} .

Since in our model gravity alone cannot cause transitions, the corrections to the transition matrix elements W_{12} and W_{21} have to vanish if the influence of the laser disappears, and this is exactly what happens in Eq. (42). We have seen this already above. It may be surprising why we have in this limit corrections at all, because for the almost freely falling atom the influence of the homogeneous gravitational field is already included in the factor of $\exp\{-itH_{c.m.}/\hbar\}$ in Eq. (38). The remaining corrections are indeed only necessary to cancel the phase factors of $\exp\{\pm i\vec{k} \cdot \vec{a}t^2/4\}$ appearing in Eq. (38).

Like the unperturbed part the corrections for small $\vec{k} \cdot \vec{a}$ are oscillating with the frequency $\hat{\omega}/2$. Furthermore, it is not difficult to see that all corrections are bounded functions of the operator $\hat{\omega}$. This enables us to study their time evolution. For $t = 0$ they vanish. For small t the corrections to the diagonal matrix elements of W grow like t^2 , and the non-diagonal corrections grow with t^3 . For moderate t this cubic dependence becomes a linear envelope of an oscillating function. The corrections are all suppressed if the Rabi frequency Ω becomes large. In this case the force caused by the laser beam is much larger than the gravitational force so that the former dominates the evolution. Note that the corrections in W_{11} and W_{22} do not vanish for large Doppler shifted detunings. This is a reasonable result since a large detuning means that the laser is out of resonance and does not affect the atomic evolution anymore. At the same time the influence of gravity is not altered so that there must remain some effect of gravity even when the detuning is large.

Another effect of gravity can be anticipated by examining W_{12} . For vanishing corrections this is a purely imaginary operator. Switching gravity on we see that it develops a real part which can become large if t grows. This is an indication that the acceleration causes additional time dependent phase factors. We will look at this more closely in the next section.

5 The evolution for long times

The results of the previous section have been obtained in discussing the weak gravity limit of the exact result. They could have also been worked out within an approximation scheme which starts with the gravitation-free case and perturbs it by a small $\vec{k} \cdot \vec{a}$. In this section we focus on a result which is of non-perturbative character: the behaviour of the atom at late times t . The formal condition which we will use is

$$|\hat{\Delta}_t| = |\hat{\Delta}_0 - \vec{k} \cdot \vec{a}t| \gg \frac{1}{\tau_a} . \quad (46)$$

In this case we can apply the asymptotic form of the functions U and V in Eq. (36) which depend on $\hat{\Delta}_t$. Since $\hat{\Delta}_t$ is an operator it would be more precise to require $|\langle \hat{\Delta}_t \rangle| \gg 1/\tau_a$ and to consider only wavepackets for which the standard deviation of $\hat{\Delta}_t$ remains sufficiently small. Because any time dependence of W_{ij} is enclosed in $\hat{\Delta}_t$ it is easy to see that the condition (46) is equivalent to $t \gg \tau_a$ (if $\hat{\Delta}_0$ is not too large). Physically this means that we consider the case when the Doppler shifted detuning becomes large if it was initially (this is $\hat{\Delta}_0$) not very large. It would be a different physical situation if the detuning is initially very large and becomes small due to the Doppler effect of the accelerated atom. This will not be considered here.

The main ingredients of the limiting process in question are explained in Appendix B. The matrix W is found to be

$$\begin{aligned}
W_{11}(\vec{p}) &\approx \sqrt{\frac{\pi}{2}} e^{-i\zeta\tau_a^2\hat{\Delta}_t^2/4} (-\zeta\tau_a\hat{\Delta}_t)^{-i\zeta\Omega^2\tau_a^2/4} e^{\pi\Omega^2\tau_a^2/16} \left\{ V\left(\frac{1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0\right) - \frac{i\zeta U(\frac{1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0)}{\Gamma(-i\zeta\Omega^2\tau_a^2/4)} \right\} \\
W_{12}(\vec{p}) &\approx -\sqrt{\frac{i\pi\Omega^2}{8\vec{k} \cdot \vec{a}}} e^{-i\zeta\tau_a^2\hat{\Delta}_t^2/4} (-\zeta\tau_a\hat{\Delta}_t)^{-i\zeta\Omega^2\tau_a^2/4} e^{\pi\Omega^2\tau_a^2/16} \left\{ V\left(\frac{-1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0\right) + \frac{4U(\frac{-1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0)}{\Omega^2\tau_a^2\Gamma(-i\zeta\Omega^2\tau_a^2/4)} \right\} \\
W_{21}(\vec{p}) &\approx \sqrt{\frac{i\Omega^2}{4\vec{k} \cdot \vec{a}}} e^{i\zeta\tau_a^2\hat{\Delta}_t^2/4} (-\zeta\tau_a\hat{\Delta}_t)^{i\zeta\Omega^2\tau_a^2/4} e^{-\pi\Omega^2\tau_a^2/16} U\left(\frac{1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0\right) \\
W_{22}(\vec{p}) &\approx e^{i\zeta\tau_a^2\hat{\Delta}_t^2/4} (-\zeta\tau_a\hat{\Delta}_t)^{i\zeta\Omega^2\tau_a^2/4} e^{-\pi\Omega^2\tau_a^2/16} U\left(\frac{-1}{2} + \theta, \frac{-i}{\sqrt{i\zeta}}\tau_a\hat{\Delta}_0\right). \tag{47}
\end{aligned}$$

The most striking feature of these operators is that (to lowest order) only their phase varies with $\hat{\Delta}_t$ and hence with t . This is not exactly true because terms containing $\hat{\Delta}_t$ are operator valued and can alter also the shape of a wavepacket. But for sufficiently narrow wavepackets in momentum space the factors containing $\hat{\Delta}_t$ in Eq. (47) simply produce an additional phase shift with logarithmic time dependence (remember that $y^{ic} = \exp(ic \ln y)$). From Eq. (38) we then can conclude that with growing time the matrix elements of the total evolution operator $U(t)$ vary also only in their phases. We will discuss this time dependence first and turn to the amplitudes thereafter.

We interpret the result as follows: Atoms exposed to gravity and a laser beam may have been in resonance with the laser at some earlier stage of their evolution. In this stage they perform a number of Rabi oscillations. But since the momentum dissipation due to spontaneous emission is neglected in our model, the atoms loose during each Rabi cycle the same amount of momentum as they gain, the net effect being zero. During the Rabi oscillations the atoms are accelerated by the earth's gravity so that their velocity increases. Because of the Doppler effect the atoms are then driven out of resonance with the laser beam so that the Rabi oscillations are vanishing with increasing time. The transition between excited and ground state is frozen. This is reflected by the fact that the absolute value of the W_{ij} is almost constant for long times.

Looking more closely to the particular matrix element U_{11} of Eq. (38) we can write the time dependent phase factor as

$$\exp \left\{ -it \left(\frac{E_e + E_g}{2\hbar} + \frac{1}{\hbar} H_{c.m} + \frac{\delta}{2} \right) \right\} \exp \left\{ \frac{it(\hat{D} - \omega_L)}{2} + \frac{i}{4} \vec{k} \cdot \vec{a} t^2 \right\} e^{-i\zeta\tau_a^2(\hat{\Delta}_t + \delta)^2/4} [-\zeta\tau_a(\hat{\Delta}_t + \delta)]^{-i\zeta\Omega^2\tau_a^2/4}. \tag{48}$$

Expanding the factor of $\hat{\Delta}_t$ and keeping only time dependent terms leads us to

$$\exp \left\{ -it \frac{E_e}{\hbar} \right\} \exp \left\{ -it \frac{1}{\hbar} H_{c.m.} \right\} [-\zeta\tau_a(\hat{\Delta}_t + \delta)]^{-i\zeta\Omega^2\tau_a^2/4}. \tag{49}$$

Here we see that each phase factor (but the last) is linear in t . The time dependence of U_{12} agrees in this limit with the one of U_{11} . This is reasonable because both matrix elements correspond to atoms which are excited at the time t . The result for U_{21} and U_{22} can be read off from Eq. (49) by replacing E_e by E_g (these matrix elements describe atoms which are in the ground state at time t) and $[-\zeta\tau_a(\hat{\Delta}_t + \delta)]^{-i\zeta\Omega^2\tau_a^2/4}$ by $[-\zeta\tau_a(\hat{\Delta}_t - \delta)]^{i\zeta\Omega^2\tau_a^2/4}$. Each factor in Eq. (49) allows a proper physical interpretation. The first exponential describes just the internal energy of the excited state. The second is the time evolution of a free point particle in a homogeneous gravitational field. Hence we see that the atom is essentially freely falling if the third term is neglected.

Nevertheless, this slowly (logarithmically) varying phase is of physical interest since it contains both Ω and \vec{a} and is therefore the only remaining time dependent effect arising from both gravity and laser light. Because it contains Ω its origin must be the Ω -term in the Hamiltonian (6) which induces transitions between ground and excited state. But since it is only a phase factor it does not describe such transitions (which would be characterized by a population transfer between $|\psi_e\rangle$ and $|\psi_g\rangle$, i.e., in a time dependent change of the absolute values of W_{ij}). Seen in this way the result seems to contradict our intuition, but this contradiction can be resolved by a comparison of the present situation with Raman transitions (see, e.g., Ref. [11]). Raman transitions are possible in a system with two lower and one upper state (" Λ system"). In this Λ system one can induce direct transitions between the two lower states without populating the upper state by applying a laser field with a large detuning versus the transition frequency between a lower and the upper state. This is similar to our case: we also have a large detuning and no population transfer to the upper state, the only difference is that we have as a form of degeneration only one lower state. Hence, we interpret the third term in Eq. (49) as a kind of Raman transition in a two-level system for which the large detuning prevents the population of the upper state during a Rabi cycle. The peculiar logarithmic time dependence of the phase is a consequence of the time dependent detuning. We should mention that because we have neglected spontaneous emission the argument does not only apply to Rabi cycles of the form $|\psi_g\rangle \rightarrow |\psi_e\rangle \rightarrow |\psi_g\rangle$ but also to cycles of the form $|\psi_e\rangle \rightarrow |\psi_g\rangle \rightarrow |\psi_e\rangle$.

We turn now to the discussion of the time independent part of W_{ij} . The phase of this part is a rapidly oscillating function of $\hat{\Delta}_0$. To get a feeling for the absolute values we consider the case of a narrow wavepacket of atoms with an initial velocity so that $\hat{\Delta}_0$ vanishes, i.e., the Doppler shifted detuning is initially zero. With the aid of the formulae in appendix B (especially Eq. (68)) it is not difficult to obtain

$$\begin{aligned} |W_{11}(\hat{\Delta}_0 = 0)| &= |W_{22}(\hat{\Delta}_0 = 0)| = \sqrt{\frac{1}{2}(1 + e^{-\pi\Omega^2\tau_a^2/4})} \\ |W_{12}(\hat{\Delta}_0 = 0)| &= |W_{21}(\hat{\Delta}_0 = 0)| = \sqrt{\frac{1}{2}(1 - e^{-\pi\Omega^2\tau_a^2/4})} \end{aligned} \quad (50)$$

This is a surprisingly simple expression with several interesting features. First, it only depends on the absolute value of $\vec{k} \cdot \vec{a}$ since the variable ζ is absent. This is reasonable since the appearance of $\zeta\Omega^2\tau_a^2$ in the exponent would allow imaginary absolute values of W_{12} and W_{21} . Second, for vanishing laser intensity ($\Omega = 0$) we find $|W_{12}| = |W_{21}| = 0$ and $|W_{11}| = |W_{22}| = 1$ in accordance with the fact that without laser beam any internal transition is impossible. For high laser intensity ($\Omega \gg 1$) we get a complete mixing: $|W_{ij}| = 1/\sqrt{2}$. This can be understood as a consequence of the large frequency of the Rabi cycles. For $\Omega \rightarrow \infty$ there is an infinite number of Rabi cycles per unit time. This is not a well defined result. But one can replace these quick oscillations by their average, and this procedure results of course in an equal probability for ground and excited state as indicated by $|W_{ij}|^2 = 1/2$. In general the transition amplitude depends on the variable $\Omega^2\tau_a^2$ which can be interpreted as representing the influence of the laser (because of Ω) acting for the time τ_a after which the atoms are essentially out of resonance.

6 The breakdown of the Magnus expansion

We now have finished the physical discussion of the atomic dynamics in a homogeneous gravitational field and a running laser wave. This last section of the paper is of purely theoretical interest. We will show here that the application of the Magnus perturbation expansion [12] would lead to unphysical results for the evolution operator. This fact was already examined in the literature (see Ref. [13] and references therein), mostly for the harmonic oscillators and simple two-level systems. A further analysis for the falling two-level atom in a running laser wave has the advantage that it is possible to use the Schrödinger picture and that we can compare the result of the Magnus expansion with the exact solution obtained above.

To perform the Magnus expansion we go back to Eq. (24) and ask whether it is possible to treat $[Q, P]$ as a small term and to apply a perturbation expansion instead of solving Eq. (24) exactly. The Magnus expansion consists of an expansion of the exponential of an operator. Setting $W = \exp(F(\lambda))$ we may try to calculate F to first order in $[Q, P]$. This can be done by using the equation (see, e.g., Ref. [14])

$$\frac{d}{d\lambda} e^{F(\lambda)} = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l!} K_l e^F \quad (51)$$

with $K_1 := dF/d\lambda$ and $K_{l+1} := [K_l, F]$ which is exactly valid for any operator F provided the exponential makes sense. Eq. (51) has the same structure as Eq. (24). The strategy to solve Eq. (24) perturbatively is therefore to find an operator F such that the sum in the r.h.s. of Eq. (51) reproduces, to first order in $[Q, P]$, just the prefactor of W in the r.h.s. of Eq. (24). In view of this prefactor we make the following guess for F :

$$F = \lambda B - \frac{1}{2}\lambda^2[Q, P]\sigma_3 + [Q, P]\sum_{n=1}^{\infty} \frac{(-1)^n \lambda^{n+2}}{(n+1)!} b_n R_n \quad (52)$$

where b_n are numbers, $R_1 := [\sigma_3, B]$, and $R_{n+1} := [R_n, B]$. Inserting this in Eq. (51) and comparing the result (to first order in $[Q, P]$) with Eq. (24) leads to the condition

$$(n+2)b_n - \frac{1}{2} + \sum_{l=0}^{n-2} \binom{n+1}{l+2} b_{n-l-1} = 0, \quad n \geq 2. \quad (53)$$

It is easy to check that the numbers b_n are related to the Bernoulli numbers B_n by $b_n = B_{n+1}$ by using the relation

$$\sum_{l=0}^{n-1} \binom{n}{l} B_l = 0. \quad (54)$$

A closed expression for F can be found by noting that

$$R_{2n} = (it\hat{\omega})^{2n-2} R_2, \quad R_{2n+1} = (it\hat{\omega})^{2n} R_1 \quad (55)$$

and $B_{2n+1} = 0$ hold. Putting everything together one finds

$$F(\lambda) = \lambda B - \frac{\lambda^2}{2}[Q, P]\sigma_3 + [Q, P]R_1 \frac{\lambda}{t^2 \hat{\omega}^2} \left\{ \frac{it\hat{\omega}\lambda}{e^{it\hat{\omega}\lambda} - 1} - 1 + \frac{it\hat{\omega}\lambda}{2} \right\} + O([Q, P]^2). \quad (56)$$

The approximate solution of our problem is obtained if λ is set to 1. It is obvious that this solution becomes singular whenever the condition

$$\hat{\omega}t = 2\pi N, \quad N \in \mathbb{N} \quad (57)$$

is fulfilled (this is possible if we neglect the decay rates γ_e and γ_g ; if they are non-zero the solution $F(\lambda)$ contains an unphysical resonance).

The approximation should be good for $||[Q, P]|| \ll 1$, i.e., for $t \ll \tau_a$. But $\hat{\omega}$ is for small detunings essentially identical to Rabi's frequency Ω . Hence the first singularity occurs at $t = 2\pi/\Omega$ which is much less than τ_a for not too small Ω . This illustrates that a perturbation approach based on the Magnus expansion fails to describe our system properly. This fact was explained as a consequence of a finite convergence radius of the Magnus expansion [13]. It may be that the reason lies in the splitting of the right hand side of Eq. (51) into one factor e^F which contains any power of $[Q, P]$, and in the sum over l which in our example is calculated only to first order in $[Q, P]$.

7 Conclusions and Outlook

In this paper we have exactly determined and discussed the time evolution of a two-level atom falling in a homogeneous gravitational field under the influence of a running laser wave. The time evolution operator has been worked out in an algebraical way. For neglected spontaneous emission the Doppler effect was shown to be the origin of the characteristic new physical features. An atom which is initially in resonance with the laser beam will first perform Rabi oscillations. The homogeneous gravitational field accelerates simultaneously the center-of-mass motion. During this acceleration the Doppler shift causes the laser frequency to drive out of resonance with the atom. Thus the Rabi oscillations are fading away until only a slow variation of the phase remains, no population transfer appears for large times. This situation is similar to Raman transitions in Λ systems.

The treatment of the model described above may be regarded as a first step towards the inclusion of the influence of gravity in situations of greater practical importance. That there is (almost) no net momentum

transfer of the laser to the atom relies on the neglect of the spontaneous emission. As long as the laser is in resonance with the atom each Rabi cycle will be completed, no total momentum transfer occurs. But if spontaneous emission is included Rabi oscillations can be incomplete. The resulting momentum transfer from the laser to the atom may be exploited to construct a gravitational atom storage if it is adjusted so that it can cancel the acceleration by the homogeneous gravitational field. The interruption of the Rabi cycles can also be managed in a coherent way, e.g., by using a three level "Λ" scheme with two hyperfine ground states. In this case a magnetic field (see, e.g., Ref. [15]) or a microwave may be used to carry the atoms back in their initial state. It should be mentioned that there is already a gravitational atom trap [16] which was theoretically studied by Wallis, Dalibard, and Cohen-Tannoudji [17]. The difference to our proposal is that we use the simultaneous action of gravity and laser forces whereas in the existing device these forces act in different time periods.

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A

The aim of this appendix is to study the behaviour of $U(\theta - 1/2, y_0 + \varepsilon)$ for small

$$\varepsilon := \sqrt{2[Q, P]} \sim \frac{t}{\tau_a} \quad (58)$$

and to derive the corresponding limiting case of W_{ij} . For brevity we have introduced the parameter $y_0 := -i\tau_a \hat{\Delta}_0 / \sqrt{i\zeta}$. We start with the observation that any derivative of $U(\theta - 1/2, y)$ with respect to y can be written in the form

$$U^{(n)}(\theta - 1/2, y) = X_n(\theta - 1/2, y)U(\theta - 1/2, y) + Y_n(\theta - 1/2, y)U'(\theta - 1/2, y) \quad (59)$$

where the prime denotes the derivative with respect to y and the functions X_n and Y_n fulfill

$$\begin{aligned} X_{n+1} &= X'_n + \left(\frac{y^2}{4} + \theta - 1/2\right) Y_n \\ Y_{n+1} &= Y'_n + X_n \end{aligned} \quad (60)$$

with $X_0 = 1$ and $Y_0 = 0$. This relation, which follows directly from the differential equations (28) and (29), will turn out to be useful for the Taylor expansion of

$$U(\theta - 1/2, y_0 + \varepsilon) = \sum_{l=0}^{\infty} \frac{\varepsilon^l}{l!} U^{(l)}(\theta - 1/2, y_0) \quad (61)$$

around y_0 . For the expansion of X_n and Y_n in ε it is of importance to note that

$$y_0 = \frac{-2P}{\varepsilon} \quad , \quad \theta = \frac{R^2}{\varepsilon^2} \quad (62)$$

holds. Bearing this in mind one can use Eq. (60) to proof by induction that

$$\begin{aligned} X_{2n} &= \varepsilon^{-2n} \left\{ (it\hat{\omega}/2)^{2n} \mp \frac{n}{2} \varepsilon^2 (it\hat{\omega}/2)^{2n-2} + O(\varepsilon^4) \right\} \\ Y_{2n} &= -P \varepsilon^{-2n+3} n(n-1) \left\{ (it\hat{\omega}/2)^{2n-4} \mp \frac{(n-2)\varepsilon^2}{2} (it\hat{\omega}/2)^{2n-6} + O(\varepsilon^4) \right\} \\ X_{2n+1} &= -P \varepsilon^{-2n+1} n^2 \left\{ (it\hat{\omega}/2)^{2n-2} \mp \frac{(n-1)\varepsilon^2}{2} (it\hat{\omega}/2)^{2n-4} + O(\varepsilon^4) \right\} \\ Y_{2n+1} &= \varepsilon^{-2n} \left\{ (it\hat{\omega}/2)^{2n} \mp \frac{n\varepsilon^2}{2} (it\hat{\omega}/2)^{2n-2} + O(\varepsilon^4) \right\} . \end{aligned} \quad (63)$$

is valid. Here we used the fact that

$$\frac{i}{2}t\hat{\omega} = \varepsilon\sqrt{\theta + \frac{y_0^2}{4}} = \sqrt{(-P)^2 + R^2} \quad (64)$$

is independent of ε . The operator $\hat{\omega}$ is defined in Eq. (43), and the upper (lower) sign in Eq. (63) holds for $U(\theta - 1/2, y_0)$ and $U(\theta + 1/2, y_0)$, respectively. Note that the same expansion holds for $V(\theta - 1/2, y_0)$ and $V(\theta + 1/2, y_0)$ simply because they are also solutions of Eqs. (28) and (29), respectively, and because these differential equations are the only relation which were used to derive Eq. (60).

We are now ready to calculate the expansion of $W_{12}(\vec{p})$ in terms of ε . We have with Eq. (36)

$$\begin{aligned} W_{12}(\vec{p}) &= \sqrt{\frac{\pi}{2}} \frac{R}{\varepsilon} \{U(\theta - 1/2, y_0)V(\theta - 1/2, y_0 + \varepsilon) - V(\theta - 1/2, y_0)U(\theta - 1/2, y_0 + \varepsilon)\} \\ &= \sqrt{\frac{\pi}{2}} R \sum_{n=0}^{\infty} \frac{\varepsilon^{n-1}}{n!} Y_n(\theta - 1/2, y_0) \{U(\theta - 1/2, y_0)V'(\theta - 1/2, y_0) - U'(\theta - 1/2, y_0)V(\theta - 1/2, y_0)\} \\ &= R \sum_{n=0}^{\infty} \frac{\varepsilon^{n-1}}{n!} Y_n(\theta - 1/2, y_0) \end{aligned} \quad (65)$$

where we have used Eq. (59) for the first step and the Wronskian relation (34) for the second step. It is now straightforward to derive a closed expression for W_{12} by exploiting Eq. (63) and the Taylor series of \cos and \sin . The result is given in Eq. (42).

The derivation of the expansion for W_{21} is similar to the previous calculations and will not be reproduced here. For W_{11} and W_{22} one further step has to be included since the functions U and V occurring in the corresponding expressions of Eq. (36) contain both the parameter $\theta - 1/2$ and $\theta + 1/2$. It is therefore necessary to make use of Eq. (35) and to deduce expressions for the W_{ii} where either only $\theta - 1/2$ or only $\theta + 1/2$ occurs. After this has been done the calculations are the same as for W_{12} . Inserting all definitions one arrives at Eq. (42).

B

In this appendix we will sketch the derivation of the long time behaviour for the operators $W_{ij}(\vec{p})$ in Eq. (36). This can be handled by using (Eq. 13.5.1 of Ref. [10])

$${}_1F_1(\alpha, \beta, z) \approx \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} e^{\pm i\pi\alpha} z^{-\alpha} + \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^z z^{\alpha-\beta} \quad (66)$$

where z is a complex number with $|z| \gg 1$, and the upper sign holds for $-\pi/2 < \arg z < 3\pi/2$ whereas the lower sign holds for $-3\pi/2 < \arg z \leq -\pi/2$. We apply this formula to the function $U(-1/2 + \theta, -i\tau_a \hat{\Delta}_t / \sqrt{i\zeta})$ and in the same manner to any other function in Eq. (36) which depends on $\hat{\Delta}_t$ by inserting it into Eqs. (32) and (33). Note that the limit depends on ζ since $\zeta = \pm 1$ determines $\arg z$. We assume $\langle -\tau_a \hat{\Delta}_t \rangle$ to be a large positive number and choose $\arg(i\zeta) = \zeta\pi/2$. Hence the "+" sign in Eq. (66) holds for $\zeta = 1$ and the "-" sign for $\zeta = -1$.

The rest of the calculation is straightforward but long. It is useful to apply $z\Gamma(z) = \Gamma(z+1)$, $\Gamma(1/2) = \sqrt{\pi}$, and

$$\begin{aligned} \Gamma(ix)\Gamma(-ix) &= |\Gamma(ix)|^2 = \pi/[x \sinh(\pi x)] \\ \Gamma(ix + 1/2)\Gamma(-ix + 1/2) &= |\Gamma(ix + 1/2)|^2 = \pi/\cosh(\pi x) \\ \Gamma(2z) &= \Gamma(z)\Gamma(z + 1/2)2^{2z-1/2}/\sqrt{2\pi} \end{aligned} \quad (67)$$

(for real x , see chapter 6 of Ref. [10]) to handle the various factors of the Γ function arising in the derivation. For the case of vanishing detuning the following equations are of use:

$$\begin{aligned} U(\alpha, 0) &= \frac{\sqrt{\pi}}{2^{\alpha/2+1/4}\Gamma(3/4 + \alpha/2)} \\ V(\alpha, 0) &= \frac{2^{\alpha/2+1/4}\sin[\pi(\alpha + 1/2)/2]}{\Gamma(3/4 - \alpha/2)}. \end{aligned} \quad (68)$$

References

- [1] C. Adams, M. Sigel, and J. Mlynek, Phys. Rep. **240**, 143 (1994).
- [2] A. Kastberg, W.D. Phillips, S.L. Rolston, R.J.C. Spreeuw, and P.S. Jessen, to appear in Phys. Rev. Lett.
- [3] J. Audretsch and K.-P. Marzlin, Phys. Rev. A **47**, 4441 (1993).
- [4] C.J. Bordé, C. Salomon, S. Avrillier, A. Van Leberghe, C. Bréant, D. Bassi and G. Scoles, Phys. Rev. A **30**, 1836 (1984).
- [5] F. Riehle, T. Kisters, A. Witte, J. Helmcke, and C.J. Bordé, Phys. Rev. Lett. **67**, 177 (1991).
- [6] U. Sterr, K. Sengstock, J. H. Müller, D. Bettermann, and W. Ertmer, Appl. Phys. B **54**, 341 (1992).
- [7] J. Audretsch and K.-P. Marzlin, J. Phys. II (France) **4**, 2073 (1994).
- [8] C.J. Bordé in *Laser spectroscopy 10*, edited by M. Ducloy, E. Giacobino, and G. Camy, World Scientific, Singapore 1992.
- [9] M. Lutzky, J. Math. Phys. **9**, 1125 (1968).
- [10] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of mathematical functions*, Dover, New York 1972.
- [11] K. Moler, D.S. Weiss, M. Kasevich, and S. Chu, Phys. Rev. A **45**, 342 (1992).
- [12] W. Magnus, Commun. Pure Appl. Math. **7**, 649 (1954).
- [13] F.M. Fernández, Phys. Rev. A **41**, 2311 (1990).
- [14] J.A. Oteo, J. Math. Phys. **32**, 419 (1991).
- [15] O. Emile, R. Kaiser, C. Gerz, H. Wallis, A. Aspect und C. Cohen-Tannoudji, J. Phys II (France) **3**, 1709 (1993).
- [16] C. Aminoff, A. Steane, P. Bouyer, P. Desbiolles, J. Dalibard, and C. Cohen-Tannoudji, Phys. Rev. Lett. **71**, 3083 (1993).
- [17] H. Wallis, J. Dalibard, and C. Cohen-Tannoudji, Appl. Phys. B **54**, 407 (1992).